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# A Version of Evans-Perkins Type Stochastic Representation Formula for Historical Superprocesses\*

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## 1 Introduction

The purpose of this article is to introduce a version of Evans-Perkins type stochastic representation formula for a generalized  $\{\gamma, a, b, g\}$ -historical superprocess (see the definition in §2). Here by Evans-Perkins type formula we mean an explicit stochastic integral representation for historical functional of a certain class, which is similar to and is a historical process counterpart of Itô-Clark formula (e.g. [U95, p.42]) in elementary stochastic calculus. The key idea of demonstration of the Itô-Clark type formula for historical superprocess is to derive a variant of stochastic integration by parts with respect to the historical process in the Perkins sense [P92].

The review of the Evans-Perkins theory [EP95] is a good point to start. There are two reasons why their integration by parts formula is so important. For one thing, it can provides with a new formula of transformations of stochastic integrals closely connected with the so-called historical processes. In addition, a generalization of formula itself is of independent interest, and it is very useful as a theoretical tool of stochastic calculus in the theory of measure-valued processes. For another, it has an extremely remarkable meaning on an applicational basis. By making use of the formula S.N. Evans and E.A. Perkins (1995) have succeeded in deriving a kind of Itô-Wiener chaos expansion for functionals of superprocesses [EP95].

S.N. Evans and E.A. Perkins have showed that any  $L^2$  functional of superprocess may be represented as a constant  $C_0$  plus a stochastic integral with respect to the associated orthogonal martingale measure  $M$  (e.g. [EP94]). Recently they have obtained the explicit representations involving multiple stochastic integrals for a quite general functional

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of the so-called Dawson-Watanabe superprocesses. Actually, the results are obtained in the setting of the historical process associated with the superprocess [EP95].

## 2 Notation and Preliminaries

Let  $C = C^d = C([0, \infty), \mathbf{R}^d)$  denote the space of  $\mathbf{R}^d$ -valued continuous paths on  $\mathbf{R}_+ = [0, \infty)$  with the compact-open topology.  $\mathcal{C} = \mathcal{B}(C)$  is its Borel  $\sigma$ -field and

$$\mathcal{C}_t = \mathcal{B}_t(C) = \sigma(y(s), s \leq t)$$

denotes its canonical filtration. For  $y, w \in C^d$  and  $s \geq 0$ , we define the stopped path by  $y^s(t) = y(t \wedge s)$  and let

$$y/s/w = \begin{cases} y(t), & \text{for } t < s, \\ w(t-s), & \text{for } t \geq s. \end{cases} \quad (1)$$

$M_F(C)$  is the space of finite measures on  $C$  with the topology of weak convergence and we define

$$M_F(C)^t := \{m \in M_F(C); y = y^t, m - a.s. y\}, \quad t \geq 0.$$

If  $P_x$  denotes Wiener measure on  $(C, \mathcal{B}(C))$  starting at  $x$ ,  $\tau \geq 0$ , and  $m \in M_F(C)^\tau$ , define  $P_{\tau, m} \in M_F(C)$  by

$$P_{\tau, m}(A) := \int_C P_{y(\tau)}(\{w; y/\tau/w \in A\}) dm(y).$$

Let

$$\Omega_H[\tau, \infty) := \{H \in C([\tau, \infty), M_F(C)); H_t \in M_F(C)^t, \forall t \geq \tau\},$$

and put  $\Omega_H := \Omega_H[0, \infty)$ . We write  $\mathcal{H}$  for the totality of Borel sets of  $\Omega_H$ . We use the notation  $H_t(\omega) = \omega(t)$  for  $\omega \in \Omega_H$  as for the canonical realization of historical process.

Fix  $0 \leq t_1 < \dots < t_n$  and  $\psi \in C_b^2(\mathbf{R}^{nd})$ . For  $y \in C$  we set

$$\begin{aligned} \bar{y}(t) &= (y(t \wedge t_1), \dots, y(t \wedge t_n)), \\ \bar{\psi}(y) &\equiv \bar{\psi}(t_1, \dots, t_n)(y) = \psi(y(t_1), \dots, y(t_n)), \end{aligned}$$

and  $\tilde{\psi}(t, y) = \bar{\psi}(y^t)$ .  $\psi_i$  (resp.  $\psi_{ij}$ ) stands for the first (resp. second) order partials  $\partial_i \psi$  (resp.  $\partial_{ij}^2 \psi$ ) of  $\psi$ .  $\nabla \bar{\psi} : [0, \infty) \times C \rightarrow \mathbf{R}^d$  is the  $(\mathcal{C}_t)$ -predictable process whose  $j$ -th component at  $(t, y)$  is given by

$$\sum_{i=0}^{n-1} \mathbf{I}(t < t_{i+1}) \psi_{id+j}(\bar{y}(t)).$$

While, for  $1 \leq i, j \leq d$ ,  $\bar{\psi}_{ij} : [0, \infty) \times C \rightarrow \mathbf{R}$  is the  $(\mathcal{C}_t)$ -predictable process defined by

$$\bar{\psi}_{ij}(t, y) := \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \mathbf{I}(t < t_{k+1} \wedge t_{l+1}) \partial_{kd+i} \partial_{ld+j}(\bar{y}(t)).$$

Let us define the domains

$$D_0 := \bigcup_{n=1}^{\infty} \left\{ \bar{\psi}(t_1, \dots, t_n); 0 \leq t_1 < \dots < t_n, \bar{\psi} \in C_0^\infty(\mathbf{R}^{nd}) \right\} \cup \{1\},$$

$$\tilde{D}_0 := \left\{ \tilde{\psi}; \tilde{\psi}(t, y) = \bar{\psi}(y^t) \text{ for some } \bar{\psi} \in D_0 \right\}.$$

Let  $\bar{\Omega} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq \tau}, \mathbf{P})$  be a filtered probability space and let  $(\omega, y) = (\omega, y_1, \dots, y_d)$  denote sample points in  $\bar{\Omega} = \Omega \times C^d$ . Here  $\tau \geq 0$  is fixed. When  $f$  is a function on  $[\tau, \infty) \times \bar{\Omega}$  taking values in a normed linear space  $(E, \|\cdot\|)$ , then a bounded  $(\mathcal{F}_t)$ -stopping time  $T$  is a reducing time for  $f$  if and only if

$$\mathbf{I}(\tau < t \leq T) \|f(t, \omega, y)\|$$

is uniformly bounded. In addition we say that a sequence  $\{T_n\}$  reduces  $f$  if and only if each  $T_n$  reduces  $f$  and  $T_n \nearrow \infty$  holds  $\mathbf{P}$ -a.s. We say that  $f$  is locally bounded if such a sequence  $\{T_n\}$  exists. We assume that

(LB)  $\gamma \in [0, \infty), a \in S^d, b \in \mathbf{R}^d$  and  $g \in \mathbf{R}$  are  $(\hat{\mathcal{F}}_t^*)$ -predictable processes on  $[\tau, \infty) \times \bar{\Omega}$  such that  $\Lambda = (\gamma, a, b, g\gamma^{-1} \mathbf{I}(\hat{g} \neq))$  is locally bounded.

Notice that the above assumption implies that  $g$  is locally bounded.

Now we introduce the martingale problem formulation of historical processes in stochastic calculus on historical trees (cf. [P92], [P95]). For  $\tau \geq 0$  and  $m \in M_F(C)^\tau$ , we define

$$A_{\tau, m} \tilde{\psi}(t, y) \equiv A(\bar{\psi})(t, y) := \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij}(t, y) \bar{\psi}_{ij}(t, y) + b(t, \omega, y) \cdot \nabla \bar{\psi}(t, y) + g(t, \omega, y) \bar{\psi}(y^t)$$

for  $\bar{\psi} \in D_0$ . We write  $\langle \mu, f \rangle$  or sometimes  $\mu(f)$  for the integral  $\int f d\mu$  when  $\mu$  is a measure and  $f$  is a suitable  $\mu$ -integrable function. Suggested by [DkTn98], we may define

**Definition 1** (cf. [P95], §2) *A predictable process  $K = \{K_t, t \geq \tau\}$  on  $\bar{\Omega}$  with sample paths a.s. in  $\Omega_H[\tau, \infty)$  is a generalized  $\{\gamma, a, b, g\}$ -historical process (GHP) (or  $(A, -\gamma\lambda^2/2)$ -historical process) if and only if  $K_t \in M_F(C)^t$  for all  $t \geq \tau$ , a.s. and  $P[K_\tau(1)] < \infty$ , and if there exists a probability measure  $\mathcal{P}$  on  $\Omega_H[\tau, \infty)$  such that it satisfies the martingale problem (MP) with initial data  $\{\tau, m\}$  and  $\{\gamma, a, b, g\}$ : for  $\forall \bar{\psi} \in D_0$ ,*

$$Z_t(\bar{\psi}) = \langle K_t, \bar{\psi} \rangle - \langle m, \bar{\psi} \rangle - \int_\tau^t \langle K_s, A(\bar{\psi})(s) \rangle ds, \quad t \geq \tau, \quad (2)$$

is a continuous  $(\mathcal{F}_t)$ -local martingale satisfying  $Z_\tau(\bar{\psi}) = 0$  and

$$\langle Z(\bar{\psi}) \rangle_t = \int_\tau^t \int \gamma(s, \omega, y) \bar{\psi}(y)^2 K_s(dy) ds, \quad \forall t \geq \tau, \quad \text{a.s.}$$

**Remark.** The existence and uniqueness of the law of  $K$  is essentially due to [F88] (cf. [DIP89]).

Set  $T_s = [s, \infty)$ , and in particular  $T_0 = [\tau, \infty)$ . Define  $C(M_F(C)) := C(T_0; M_F(C))$ , and we write  $C(t) = (\tau, t] \times C$  for the integral domain. When  $\mathcal{F}$  is the  $\sigma$ -field or the usual filtration, then  $f \in \mathcal{F}$  indicates that the function  $f$  is  $\mathcal{F}$ -measurable and  $\mathcal{P}(\mathcal{F})$  is the totality of  $(\mathcal{F})$ -predictable functions, and  $b\mathcal{P}(\mathcal{F})$  denotes the whole space of functions that are all bounded elements of  $\mathcal{P}(\mathcal{F})$ . We use the symbol  $U(M_F(C))$  for an admissible subset of the space  $C(C(M_F(C)); \mathbf{R})$ ; more precisely  $U(M_F(C))$  is the totality of real valued continuous functions  $F$  on  $C(M_F(C))$  such that for some compactly supported finite measure  $L(dt)$  on  $T_0$ , the estimate

$$|\Delta F(h, g)| \leq \int_{T_0} g(t, C) L(dt)$$

holds for all  $h, g \in C(M_F(C))$ , where we define  $\Delta F(x, y) := F(x + y) - F(x)$ .

### 3 Predictable Representation Property

Let  $\{T_N\}$  be a reducing sequence. Take a sequence  $\{\bar{\psi}_n\}$ ,  $\bar{\psi}_n \in D_0$  such that  $\bar{\psi}_n$  converges bounded pointwise (*bp* for short) to  $\psi$ , namely,

$$\bar{\psi}_n \rightarrow \psi, \quad bp \quad (n \rightarrow \infty).$$

An application of dominated convergence theroem together with the local boundedness of  $\gamma$  implies that

$$\langle Z(\bar{\psi}_n - \bar{\psi}_m) \rangle_t \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

for  $\forall t \geq \tau$ , a.s. Therefore we obtain

**Proposition 1** *There is an a.s. continuous adapted process  $\{Z_t(\psi); t \geq \tau\}$  such that*

$$\sup_{\tau \leq t \leq N} |Z_t(\bar{\psi}_n) - Z_t(\psi)| \rightarrow 0$$

*holds in probability ( w.r.t.  $P$  ) as  $n \rightarrow \infty$  for  $\forall N > \tau$ .*

To proceed our discussion, we need the following lemmas.

**Lemma 1** (cf. Corollary 2.2, p.11, [P95]) *Let  $T$  be a reducing time for  $(\gamma, g)$ . Then we have*

$$(a) \quad 0 < P[K_T(1)] \leq P[\sup_{\tau \leq t \leq T} |K_t(1)| + \langle Z(1) \rangle_T] < \infty.$$

(b) *If  $P[K_\tau(1)^p] < \infty$  for  $p \in N$ , then*

$$P \left\{ \left( \sup_{\tau \leq t \leq T} |K_t(1)| \right)^p + \langle Z(1) \rangle_T^p \right\} < \infty.$$

**Lemma 2** (cf. [EP94, p.123])  *$D_0$  is dense in  $b\mathcal{B}(C)$  relative to the bounded pointwise convergence topology.*

We may use Lemma 1 to obtain

$$\sup_{\tau \leq t \leq T_N} |Z_t(\bar{\psi}_n) - Z_t(\psi)| \rightarrow 0 \quad \text{in } L^2$$

as  $n \rightarrow \infty$ , for  $\forall N \in \mathbb{N}$ . Clearly  $Z_t(\psi)$  is a continuous  $(\mathcal{F}_t)$ -local martingale whose quadratic variation process is given by

$$\langle Z(\psi) \rangle_t = \int_{\tau}^t \int_C \gamma(s, \omega, y) \psi(y)^2 K_s(dy) ds. \quad (3)$$

By virtue of Lemma 2, it is a routine work to show that this  $Z_t$  extends to an orthogonal martingale measure

$$\{Z_t(\psi); t \geq \tau, \psi \in b\mathcal{B}(C)\}.$$

Consequently, the mapping  $t \mapsto Z_t(\psi)$  is a continuous local martingale satisfying Eq.(3) for each  $\psi \in b\mathcal{B}(C)$ , and  $\psi \mapsto Z_{t \wedge T_N}(\psi)$  is an  $L^2$ -valued measure on  $\mathcal{B}(C)$  for each  $t \geq \tau$ ,  $N \in \mathbb{N}$ . By a trivial localization argument, we may define the stochastic integral

$$Z_t(\psi) = \int_{\tau}^t \int \psi(s, \omega, y) dM(s, y) \quad (4)$$

( $\exists$  an orthogonal martingale measure  $M = M^K$  in the sense of Walsh [W86, Chapter 2] ) such that

$$\langle Z(\psi) \rangle_t = \int_{\tau}^t \langle K_s, \gamma(s, \omega) \psi(s, \omega)^2 \rangle ds, \quad (5)$$

$\forall t \geq \tau$ , a.s., as long as  $\psi$  belongs to  $L_{loc}^2(K, \mathbf{P})$ . Here  $L_{loc}^2(K, \mathbf{P})$  denotes the  $L^2$  space of  $(\mathcal{F}_t \times C)_{t \geq \tau}$ -predictable functions  $f$  and

$$\int_{\tau}^t \int \gamma(s, y) f(s, y)^2 K_s(dy) ds < \infty$$

for  $\forall t \geq \tau$ ,  $\mathbf{P}$ -a.s.

We write  $f \in L^2(K, \mathbf{P})$  (resp.  $L_{\infty}^2(K, \mathbf{P})$ ) if, in addition,

$$\mathbf{P} \left\{ \int_{\tau}^t \int \gamma(s, \omega, y) f(s, \omega, y)^2 K_s(dy) ds \right\} < \infty, \quad \forall t > 0,$$

respectively,

$$\mathbf{P} \left\{ \int_{\tau}^{\infty} \int \gamma(s, \omega, y) f(s, \omega, y)^2 K_s(dy) ds \right\} < \infty.$$

**Theorem 1 (Predictable Representation Property)** *If  $V \in L^2(\Omega, \mathcal{F}, P)$ , then there is an  $f$  in  $L_{\infty}^2(K, P)$  such that*

$$V = P[V] + \int_{\tau}^{\infty} \int f(s, \omega, y) dM^K(s, y), \quad P - a.s. \quad (6)$$

The proof of Theorem 1 will be given in the succeeding section.

*Remark.* The predictable representation property was proved by Evans-Perkins (1994) [EP94, Theorem 1.1] for the  $(Y, -\lambda^2/2)$ -superprocess with a Hunt process  $Y$  as its underlying process. In [EP95] a variant of the stochastic integral representation formula of the above type was proved for the  $(Y, -\lambda^2/2)$ -historical process with a Markov process  $Y$ .

## 4 Proof of Theorem 1

If  $f \in b\mathcal{B}(C)$ , then the moment  $\mathbf{P}[K_t(f)]$  is uniformly bounded as  $t$  ranges over a compact subset of  $[\tau, \infty)$ . We have the following explicit formula for the moment, namely,

**Lemma 3**  $P[K_t(f)] = P_{\tau, \nu}[f(Y^t)]$  holds for every  $f$  in  $b\mathcal{B}(C)$  under  $\nu \in M_F(C)^\tau$ , where  $Y^t$  is the corresponding stopped path-valued process.

We set  $\hat{E} := \{(s, y) \in [\tau, \infty) \times C; y^s = y\}$  and define a measure  $Q_{s, y}$  on  $(C, \mathcal{C})$  by

$$Q_{s, y}(A) := P_{y(s)}\{w \in C; (y/s/w) \in A\}, \quad A \in \mathcal{C}, \quad (s, y) \in \hat{E}.$$

Then a similar argument as in [F88] (cf. Theorem 2.1.3, [DP91]) allows us to show

**Proposition 2** Assume that  $T_{s, t}f(y) := P_{y(s)}[f(y/s/Y^{t-s})]$  satisfies the semigroup property for  $(s, y) \in \hat{E}$ ,  $t \geq s$ , and  $f \in b\mathcal{B}(C)$ . Then we have

$$P[\exp\{-\langle K_t, f \rangle\}] = \exp\{-\langle m, V_{\tau, t}f \rangle\},$$

for all  $f \in b\mathcal{B}(C)$  and  $m \in M_F(C)$ . Moreover,  $\{V_{\tau, t}\}$  forms a semigroup on  $b\mathcal{B}(C)$ , and  $V_{s, t}f(y) \equiv v_{s, t}(y)$  is Borel measurable as a function of  $(s, y, t)$  in  $\hat{E} \times [\tau, \infty)$  with  $t \geq s$ , and is the unique solution of

$$v_{s, t}(y) = P_{y(s)}[f(y/s/Y^{t-s})] - \frac{1}{2} \int_{\tau}^{t-s} P_{y(s)}[\gamma(u, \omega, y)v_{u+s, t}(y/s/Y^u)] du.$$

*Proof of Lemma 3.* According to the same discussion as in Theorem 2.1.5 [DP91, p.19], we can deduce from Proposition 2 that under  $\nu \in M_F(C)^\tau$

$$\mathbf{P}[\langle K_t, f \rangle] = \langle \nu, G_{\tau, t}f \rangle, \quad \text{--- -- -- -- --} (*)$$

where  $G_{s, t}f(y) = Q_{s, y}[f(Y_t)]$ . A simple computation reads

$$\begin{aligned} \langle \nu, G_{\tau, t}f \rangle &= \int_C Q_{\tau, y}[f(Y_t)] \nu(dy) \\ &= \int_C \left\{ \int_{C^t} f(Y_t) P_{y(\tau)}\{w \in C; (y/\tau/w) \in d\zeta\} \right\} \nu(dy) \\ &= \int_{C^t} f(Y^t) \int_C P_{y(\tau)}\{(y/\tau/Y) \in d\zeta\} \nu(dy) \\ &= \int_{C^t} f(Y^t) P_{\tau, \nu}(dy) = P_{\tau, \nu}[f(Y^t)], \end{aligned}$$

because we made use of the Fubini theorem in the second line. By (\*), this concludes the proof. Q.E.D.

Suggested by the argument [MP92, pp.331-332] (also see [EP95, pp.1779-1780]), we define

$$F^{\tau, \nu} := \{\varphi \in b(\mathcal{B}([\tau, \infty)) \times C); \varphi(t, y) = \varphi(t, y^t) \text{ for all } t \geq \tau, \\ \text{the map } t \mapsto \varphi(t, Y) \text{ is } P_{\tau, \nu} - \text{a.s. right continuous, } \forall t \geq \tau\}$$

under  $\nu \in M_F(C)^t$ , and  $\tilde{F}^{\tau,\nu}$  is the set of bounded functions  $\psi$  in  $\mathcal{B}([\tau, \infty)) \times \mathcal{F} \times \mathcal{C}$  such that

$$\psi(\cdot, \omega, \cdot) \in F^{\tau,\nu}, \quad P - \text{a.s.},$$

and the condition (C) is compatible with the definition of  $K$  in §2.

(C) For  $H_t \in M_F(C)^t$ ,  $\mathbf{P}$ -a.s. for all  $t \geq \tau$  with  $Y$  as its corresponding path-valued process, and for all  $\varphi \in F^{\tau,\nu}$ ,

$$M_t(\varphi) := \langle H_t, \varphi(t, \cdot) \rangle - \langle \nu, \varphi(\tau, \cdot) \rangle - \int_{(\tau, t]} \langle H_s, \psi(s, \omega, \cdot) \rangle ds, \quad t \geq \tau, \quad \text{under } \nu \in M_F(C)^\tau,$$

is a continuous  $(\mathcal{F}_t)_{t \geq \tau}$  martingale for which  $M_\tau(\varphi) = 0$  and

$$\langle M(\varphi) \rangle_t = \int_{(\tau, t]} \int_C \gamma(s, \omega, y) \varphi(s, y)^2 H_s(dy) ds.$$

Let  $A^{\tau,\nu}$  denote the set of pairs  $(\varphi, \psi)$  in  $F^{\tau,\nu} \times \tilde{F}^{\tau,\nu}$  such that

$$Z_t := \varphi(t, Y) - \varphi(\tau, Y) - \int_{(\tau, t]} \psi(s, Y) ds, \quad t \geq \tau,$$

is a  $(\bar{\mathcal{C}}_t^\nu)_{t \geq \tau}$ -martingale under  $P_{\tau,\nu}$ , where  $\bar{\mathcal{C}}_t^\nu$  is the  $\sigma$ -field generated by  $\mathcal{C}_{t+}$  and the  $P_{\tau,\nu}$ -null sets in  $C$ .

**Proposition 3** *There exists for each  $n \in N$  a function  $g_n = g_n(t, \omega, y)$  in  $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$  such that*

$$V = P[V] + \lim_{n \rightarrow \infty} \int_{(\tau, \infty)} \int_C g_n(s, \omega, y) dM^K(s, y),$$

with  $L^2(P)$ -convergence.

*Proof.* Recall the condition (C). By virtue of Theorem 2 and Proposition 2 of Jacod (1977) [J77] (e.g. [EP94, p.124] or [EP95, p.1796]), we can deduce that for each  $n \in N$  there exist suitable pairs

$$(\varphi_n^1, \psi_n^1), \dots, (\varphi_n^{N(n)}, \psi_n^{N(n)}) \in A^{\tau, m},$$

(relative to  $K_t$ ),  $\xi_n^1, \dots, \xi_n^{N(n)} \in b\mathcal{P}(\mathcal{F}_t)$ , and  $\{t_n\}_n \subset (\tau, \infty)$  such that  $t_n \nearrow \infty$  (as  $n \rightarrow \infty$ ) and

$$V = \mathbf{P}[V] + \lim_{n \rightarrow \infty} \int_{(\tau, t_n]} \int_C \sum_k \xi_n^k(s, \omega) \varphi_n^k(s, y) dM^K(s, y),$$

where the convergence is in  $L^2(\mathbf{P})$ . Moreover, we can choose a bounded  $(\mathcal{C}_t)_{t \geq \tau}$ -predictable function  $\eta$  such that

$$\int \int_{C(t)} \xi(s, \omega) \varphi(s, y) dM^K(s, y) = \int \int_{C(t)} \xi(s, \omega) \eta(s, y) dM^K(s, y), \quad P - \text{a.s.}, \quad \forall t \geq \tau,$$



for each  $(\varphi, \psi) \in A^{\tau, m}$  and each  $\xi$  in  $b\mathcal{P}(\mathcal{F}_t)$ , and also that the  $y$ -section

$$\{(s, y) \in [\tau, \infty) \times C; \varphi(s, y) \neq \eta(s, y)\}$$

is a countable set. By the property of stochastic integral and the Fubini type theorem, we readily obtain

$$\begin{aligned} P \left| \int \int_{C(t)} \xi \varphi dM^K - \int \int_{C(t)} \xi \eta dM^K \right|^2 &= P \left[ \int \int_{C(t)} \gamma \xi(s)^2 \{\varphi(s, y) - \eta(s, y)\}^2 dK_s ds \right] \\ &\leq C_0 \cdot P \left[ \int_{\tau}^t \int_C \{\varphi(s, y) - \eta(s, y)\}^2 K_s(dy) ds \right] \\ &= C_0 \int_{\tau}^t P [K_s(|\varphi_s - \eta_s|^2)] ds. \end{aligned}$$

for some constant  $C_0$ . By Lemma 3, the last term in the above can be replaced by

$$\int_{\tau}^t P_{\tau, m} [|\varphi(s, Y^s) - \eta(s, Y^s)|^2] ds,$$

which, indeed, becomes null if we apply the Fubini theorem again because we employed the condition

$$\int_{(\tau, t]} \{\varphi(s, Y) - \eta(s, Y)\}^2 ds = 0, \quad \forall t > \tau, \quad P_{\tau, m} - a.s.$$

So that, by making use of the above-mentioned  $\eta$ , we have only to set

$$g_n(s, \omega, y) = \sum_k \xi_n^i(s, \omega) \eta_n^i(s, y)$$

for each  $n$ . This completes the proof. Q.E.D.

By virtue of the arguments in the proof of Proposition 3, we have that

$$\begin{aligned} 0 &= \lim_{n, k \rightarrow \infty} \left\| \int \int_{C(\infty)} g_n(s, \cdot, y) dM^K(s, y) - \int \int_{C(\infty)} g_k(s, \cdot, y) dM^K(s, y) \right\|_{L^2(P)}^2 \\ &= \lim_{n, k \rightarrow \infty} \|g_n(\cdot) - g_k(\cdot)\|_{L_{\infty}^2(K, P)}^2. \end{aligned}$$

Hence there exists a limit function  $f$  in  $L_{\infty}^2(K, P)$  such that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} P \left[ \int_{\tau}^{\infty} \int_C \gamma(s, \omega, y) \{g_n(s, \omega, y) - f(s, \omega, y)\}^2 K_s(dy) ds \right] \\ &= \lim_{n \rightarrow \infty} P \left| \int \int_{C(\infty)} g_n(s, \omega, y) dM^K(s, y) - \int \int_{C(\infty)} f(s, \omega, y) dM^K(s, y) \right|^2. \end{aligned}$$

Immediately this implies from Proposition 3 that

$$\begin{aligned} V &= P[V] + \lim_{n \rightarrow \infty} \int \int_{C(\infty)} g_n(s, \omega, y) dM^K(s, y) \\ &= \int \int_{C(\infty)} f(s, \omega, y) dM^K(s, y). \end{aligned}$$

which completes the proof of Theorem 1.

## 5 Canonical Measure and Campbell Measure

For  $y \in D = D(\mathbf{R}_+; \mathbf{R}^d)$ , we define  $y^{t-}(s)$  as  $y(s)$  itself if  $s < t$  and as  $y(t-)$  if  $s \geq t$ .  $Q(s, y)$  is a  $\sigma$ -finite measure on  $C(M_F(D))$  such that

$$Q\left(s, y^{s-}; \{h \in C(M_F(D)); \tau \leq \exists t \leq s, h(t) \neq 0\}\right) = 0,$$

which can be defined by the canonical measure  $R(\tau, t, y; d\zeta)$  [D93] associated with the law of  $K_t = K(t)$  and the path restriction mapping  $\pi$  (cf. §2, pp.1781-1782 in [EP95]) together with a discussion involved with the Dawson-Perkins theory(1991) (e.g. Theorem 2.2.3(pp.27-28) and Proposition 3.3(pp.38-39) in [DP91]). Here  $R$  is characterized by

$$\log \mathcal{P}_{s, \delta_y}[\exp\langle K_t, -\varphi \rangle] = \int_{M_F(M_F(C))} \left(e^{-\langle \zeta, \varphi \rangle} - 1\right) R(s, t, y; d\zeta)$$

(cf. Lemma 1 in [Dk99c]; see also [DP91, Proposition 3.3, pp.38-39]). Let  $F$  be a real valued Borel function on  $C(M_F(C))$ . Assume that

$$I_{s,y}^Q[\Delta F](h) := \int_{C(M_F(C))} \Delta F(h, g) Q(s, y^{s-}; dg) \quad (7)$$

is well-defined and bounded below for all  $s > \tau$ ,  $y \in C$ , and  $h \in C(M_F(C))$ . For a bounded  $(\mathcal{F}_t)$ -stopping time  $T$ , we define the Campbell measure  $P_T$  associated with  $K(t)$  by

$$P_T(A \times B) := \mathbf{P}(K(T, A) \cdot \mathbf{I}_B\{K(T)\})/m(C) \quad (8)$$

for any  $A \times B \in (C \times \Omega, \mathcal{C} \times \mathcal{F})$  (cf. [P95], p.21; or [DP91], p.62). Notice that  $K_\tau = m$ . Since the mapping  $(s, y, \omega) \mapsto I_{s,y}^Q[\Delta F](K(\omega))$  is bounded below and measurable with respect to the product of the predictable  $\sigma$ -field associated with the filtration  $(\mathcal{C}_t)$  and the  $\sigma$ -field  $\mathcal{F}$ , we can apply Lemma 2.2(p.1783) [EP95] together with the projection operation argument and the predictable section theorem (e.g. Theorem 2.14(p.19) or Theorem 2.28(p.23), [JS87]; see also [E82], pp.50-52), to deduce that there exists a  $(\mathcal{C}_t \times \mathcal{F}_t)_{t \geq \tau}$ -predictable function  $Pr[F](s, y, \omega) : (\tau, \infty) \times C \times \Omega \rightarrow \mathbf{R}$  such that

$$P_T\{I^Q[\Delta F](T)/(C \times \mathcal{F})_T\} = Pr[F](T, \omega, y) \quad (9)$$

holds  $P_T$ -a.s. for all bounded  $(\mathcal{F}_t)$ -predictable stopping times  $T > s$ . It is quite interesting to note that in particular

$$\mathbf{P} \int_C I^Q[\Delta F](T, y) K(T, dy) = \mathbf{P} \int_C Pr[F](T, y) K(T, dy).$$

We shall introduce an approximation map. For each  $l \in \mathbf{N}$ , let us choose a partition  $\Delta(l) = \{t^{(l)}(j); 1 \leq j \leq k[l]\}$  such that  $\tau = t^{(l)}(0) < t^{(l)}(1) < \dots < t^{(l)}(k[l]) < \infty$ ,

$$\lim_{l \rightarrow \infty} \sup_k \Delta t[l; k] = 0 \quad \text{and} \quad \lim_{l \rightarrow \infty} t^{(l)}(k[l]) = +\infty.$$

The approximation map  $W[l]$  from  $C(M_F(C))$  into  $C(M_F(C))$  is defined by

$$W[l](g)(t) := \{Sb(t^{(l)}(i+1)) \cdot g(t^{(l)}(i)) - Sb(t^{(l)}(i)) \cdot g(t^{(l)}(i+1))\} \Delta t[l; i]^{-1}$$

if  $t \in [t^{(l)}(i), t^{(l)}(i+1))$ , and  $:= g(t^{(l)}(k[l]))$  if  $t \geq t^{(l)}(k[l])$ , for any element  $g$  of  $C(M_F(C))$  with  $Sb(k) = k - t$ . Immediately we get

**Lemma 4** (cf. Lemma 4, [DK98a]) *Let  $F$  be an element of  $C(C(M_F(C)); R)$ . Then for all  $g \in C(M_F(C))$*

$$\lim_{l \rightarrow \infty} (F \circ W[l])(g) = F(g).$$

## 6 Random Measures and Assumptions

We shall introduce the assumptions for our main results (Theorem 2, Theorem 3 and Theorem 4) which are stated in the succeeding section.  $C^t$  denotes the image of  $C$  under the map:  $y \mapsto y^t$ . We define a measure  $K^*[s, t]$  on  $C^s$  by  $K^*[s, t](F) := K_t(\{y : y^s \in F\})$ . Then the measure  $K^*[s, t]$  is atomic with a finite set of atoms, and we write  $L[s, t](\subset C^s)$  for the locations of these atoms. For  $s \in (a, b]$ , let  $\lambda_s[\varphi]$  be the random measure on  $C$  that places mass  $\varphi(s, y)$  at each point  $y$  in  $(L[b, c])^s = L[s, c]$ . With some localization arguments in stochastic calculus, the Perkins-Girsanov theorem of Dawson type [P95] guarantees the existence of a probability measure  $\mathbf{Q}_N$  on  $(\Omega, \mathcal{F})$  such that

$$\begin{aligned} \left. \frac{d\mathbf{Q}_N}{d\mathbf{P}} \right|_{\mathcal{F}_t} = \exp \left\{ \int_{\tau}^{t \wedge T_N} \int g \gamma^{-1}(s) \mathbf{I}(g(s) \neq 0) dM^K(s, y) \right. \\ \left. - \frac{1}{2} \int_{\tau}^{t \wedge T_N} \int g^2 \gamma^{-1}(s) \mathbf{I}(g(s) \neq 0) K_s(dy) ds \right\}. \end{aligned}$$

For brevity's sake we rather write  $\mathcal{E}(t \wedge T_N)$  than the above. On this account,  $K_{\cdot \wedge T_N}$  satisfies the martingale problem (MP) $[\gamma_N, a_N, b_N, 0]$  instead of (MP) $[\gamma, a, b, g]$ , where we set  $f_N := f \cdot \mathbf{I}(\tau < t \leq T_N)$ . Moreover, for  $s \in (a, b]$ ,  $y \in C^s$ , the symbol  $\mathcal{M}[s, y]$  denotes the mapping of the set of functions  $\{m : (\tau, \infty) \rightarrow M_F(C)\}$  into itself and is defined as follows: i.e.,  $\{\mathcal{M}[s, y]m\}_t(F)$  is equal to  $m_t(F)$  if  $t < s$ , or is equal to  $m_t(\{y' \in F : (y')^s \neq y\})$  if  $t \geq s$ .

Let us now introduce assumptions for our principal results.

(A.1)  $g : [\tau, \infty) \times \Omega \times C \rightarrow \mathbf{R}$  is a  $(\mathcal{F}_t \times \mathcal{C}_t)^*$ -predictable process such that  $g\gamma^{-1} \cdot \mathbf{I}(g \neq 0)$  is locally bounded.

(A.2) For any predictable function  $f$  on  $[\tau, \infty) \times I \times C^* \times \Omega$ , the counting measure  $n^*$  satisfies

$$\mathbf{P} \int_{C^*} n^*((s, t] \times I) G_t(dx) = m(C^*)(t - s)$$

where  $G_t$  is a marked historical process corresponding to  $K$  and  $N_t$  is the martingale measure associated with  $G_t$  (cf. §7 for details).

(A.3) There exists a random measure  $\Lambda_\varphi$  on  $(\tau, \infty) \times C$  such that

$$\int \int_{C(\infty)} f(s, y) \Lambda_\varphi(ds \otimes dy) = \int_{a+}^b \int_C f(s, y) \lambda_s[\varphi](dy) ds$$

holds for any suitable predictable function  $f$ .

(A.4)  $\Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1}$  is uniformly bounded in  $s$ ,  $K_s$ -a.e.  $y$ ,  $\mathbf{Q}_N$ -a.s.

(A.5) There exists some constant  $C_0 (> 0)$  such that

$$\int \int_{C(t)} \Psi(s, y)^2 \mathcal{E}(t \wedge T_N)^{-2} \gamma(s, y) K_s(dy) ds \leq C_0$$

holds  $\mathbf{Q}_N$ -a.s., for all  $t \geq \tau$ .

Note that we shall assume (A.1)-(A.5) hereafter all through the whole paper.

## 7 Stochastic Integration Formulae : Main Results

The followings are our main results in this paper. The first one is a finite dimensional version of Evans-Perkins type stochastic integration by parts formula. Let  $K$  be a predictable measure-valued process whose law is specified by a general martingale problem (MP) $[\tau, K_\tau, \gamma, a, b, g]$ .

**Theorem 2** (cf. [Dk98b]) *Assume that  $\Phi : C(M_F(C)) \rightarrow R$  is a cylinder function with bounded representing function  $\varphi : [M(C)]^k \rightarrow R$  and base  $\tau < t(1) < \dots < t(k)$ , such that*

$$|\Delta\varphi(\alpha, \beta)| \leq c_0 \sum_j \beta_j(C)$$

*for some positive constant  $c_0$ , for all  $\alpha, \beta = (\beta_j) \in [M(C)]^k$ . Then for  $t > \tau$*

$$P \left\{ \Phi(K) \int \int_{C(t)} \Psi(s, y) dM^K(s, y) \right\} = P \int \int_{C(t)} Pr[\Phi](s, y) \Psi(s, y) \gamma(s, y) K_s(dy) ds$$

*holds where  $\Psi$  is a bounded  $(\mathcal{C}_t \times \mathcal{F}_t)_{t \geq \tau}$ -predictable function,  $K_t$  is a GHP, and  $Pr[\Phi]$  is a predictable function determined by (9) in accordance with the given  $\Phi$ .*

**Remark 1.** The assertion of the above theorem is quite similar to Theorem 2.4(p.1785, §2, [EP95]).

**Theorem 3 (Stochastic Integration By Parts)** *Let  $F \in U(M_F(C))$ . If  $\Psi$  is an element of  $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$ , then for all  $t > s$ ,*

$$\begin{aligned} P \left\{ F(K) \int \int_{C(t)} \Psi(s, y) dM^K(s, y) \right\} \\ = P \int \int_{C(t)} Pr[F](s, y) \gamma(s, y) \Psi(s, y) K_s(dy) ds. \end{aligned} \quad (10)$$

*Remark 2.* Note that it is not hard to extend the assertion in Theorem 2 to the case of a more general functional  $F(K)$ . As a matter of fact, once the integral formula as given in Theorem 2 is established, it is a kind of routine work to generalize it (cf. §3, [Dk98a]). We shall refer to this generalization in §9.

**Theorem 4 (A Variant of Evans-Perkins Type Formula)** *Let  $F \in U(M_F(C))$ .*

$$F(K) = P[F(K)] + \int_{\tau+}^{\infty} \int Pr[F](s, y) dM^K(s, y) \quad (11)$$

where  $Pr[F](s, y)$  is a  $\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$ -measurable version (relative to  $P_T$ ) of

$$P_T \left[ \int_{C(M_F(C))} \Delta F(K, h) Q(s, y^{s-}; dh) / (\mathcal{D} \times \mathcal{F})_T \right].$$

## 8 Marked Historical Processes and Girsanov-Dawson-Perkins Theorem

Set  $I = [0, 1]$ ,  $E^* = C \times I$  and  $C^* = C(\mathbf{R}_+, E^*)$ , and let  $\mathcal{C}^*$  (resp.  $\mathcal{C}_t^*$ ) be the Borel  $\sigma$ -field (resp. the canonical filtration) of  $C^*$ . Put  $x = (y, n) \in E^*$ . Let  $G$  be the corresponding counterpart historical process of  $K$  starting at  $(\tau, \mu)$ , defined on the stochastic basis  $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbf{P}^*)$ . Suppose that  $\varphi : (\tau, \infty) \times C \times \Omega \rightarrow I$  be an element of  $\mathcal{P}(\mathcal{C}_t \times \mathcal{H}_t)$ . Given any cadlag function  $n : \mathbf{R}_+ \rightarrow I$ , we can construct a  $\sigma$ -finite counting measure  $n^*$  on  $\mathbf{R}_+ \times I$  by assigning an atom of mass one to each point  $(s, z)$  such that  $n(s) - n(s-) = z \neq 0$ . Put

$$A(t, x, \omega) := n^*(\{(s, z) \in [\tau, t] \times I; \varphi(s, y, \omega) > z\}) \quad (12)$$

and  $B(t, x, \omega) = \mathbf{I}\{A(t, x, \omega) = 0\}$ . Then we can define an  $M_F(C)$ -valued process  $K[\varphi](t)$  by

$$K[\varphi; J](t) := \int_{C^*} \mathbf{I}\{J\}(y) B(t, x) G_t(dx). \quad (13)$$

Put

$$I_1(\varphi, N) = \int \int_{C^*(t)} \varphi(s, y) dN(s, x), \quad \text{and} \quad I_2(\varphi, G) = \int \int_{C^*(t)} \gamma(s, y) \varphi(s, y)^2 G_s(dx) ds$$

with  $C^*(t) = (\tau, t] \times C^*$ . Then we define

$$\Lambda[\varphi](t) := \exp \left\{ I_1(\varphi, N) - \frac{1}{2} I_2(\varphi, G) \right\}. \quad (14)$$

Note that  $\Lambda[\varphi](t)$  is a  $\mathcal{H}_t$ -martingale. The new probability space  $(\Omega, \mathcal{H}, \mathbf{P}^*[\varphi])$  is defined by  $\mathbf{P}^*[\varphi]\{F\} := \mathbf{P}^*\{F \cdot \Lambda[\varphi](t)\}$  (cf. [Dk98a]) for any  $F \in b\mathcal{H}_t$  with

$$\mathcal{H} := \bigvee_{t \geq \tau} \mathcal{H}_t \quad (15)$$

(see Theorem 2.1(pp.125-126) and Theorem 2.3b(p.127), [EP94]). It is easy to show the following proposition if we apply Dawson's Girsanov theorem [D93] (see also [P95]).

**Proposition 4** (cf. Theorem 5.1, p.1798, [EP95]) *The law of  $K[\varphi]$  under  $P[\varphi]$  is equivalent to the law of  $K$  under  $P$ .*

## 9 Sketch of Proofs of Main Theorems

### §9.1 Generalization of the Cylinder Function Case: Proof of Theorem 3

As mentioned in Remark 2 of §7, the essential part of an extension of the Evans-Perkins type integration formula is compressed into the study on its finite dimensional case, namely, Theorem 2. The general case easily follows from a kind of routine work [Dk98a]. We define a real valued function  $L^*$  on  $C(M_F(C))$  by

$$L^*[g] := \int_{T_0} g(t, C) L(dt) = \langle L, g(\cdot, C) \rangle. \quad (16)$$

In connection with the measure  $L$  (see §2), we introduce the finite measure  $L(l) \equiv L(l, dt)$  which concentrates its mass on  $\{t^{(l)}(j); 0 \leq j \leq k[l]\}$  (cf. [Dk98a, p.5]). We have  $(L^* \circ W[l])[g] = \langle L(l), g(\cdot, C) \rangle$  for  $g \in C(M_F(C))$ . Recall that

$$\int g(t, C) Q(s, y; dg) = \int \xi(C) R(s, t, y; d\xi) = 1$$

holds (cf. Lemma 3, [Dk99a]) with ease for  $s < t$  from Lemma 3.4(pp.41-43), [DP91]. Then it is easy to verify the followings:

$$\mathbf{P} \int \int_{C(t)} \{Q(s, y^{s-}) L^*[g]\} K_s(dy) ds = \lim_{l \rightarrow \infty} \mathbf{P} \int \int_{C(t)} \{Q(s, y^{s-}) (L^* \circ W[l])[g]\} K_s(dy) ds$$

holds with  $g \in C(M_F(C))$  for all  $t > \tau$ , and

$$\begin{aligned} & \mathbf{P} \int \int_{C(t)} Pr[F](s, y) Z(s, y) K_s(dy) ds \\ &= \lim_{l \rightarrow \infty} \mathbf{P} \int \int_{C(t)} Pr[F \circ W[l]](s, y) Z(s, y) K_s(dy) ds. \end{aligned} \quad (17)$$

holds for all  $t > \tau$  if  $Z \in \mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$ . Since, for each  $n \geq 1$ ,  $\mathbf{P}\{K_t(C)^n\}$  is uniformly bounded on compact intervals, we can readily deduce that  $\mathbf{P}\{(L^* \circ W[l])[K]^n\}$  is bounded in  $l$  for each  $n \geq 1$ . Moreover,

$$\mathbf{P} \left\{ F(K) \int \int_{C(t)} \Psi(s, y) dM(s, y) \right\} = \lim_{l \rightarrow \infty} \mathbf{P} \left\{ (F \circ W[l])(K) \int \int_{C(t)} \Psi(s, y) dM(s, y) \right\}.$$

To complete the extension discussion in this section we have only to observe that  $F \circ W[l]$  satisfies all the conditions of Theorem 2 (cf. Lemma 22, pp.9-10, [Dk98a]). Thus we have a finite dimensional special case of stochastic integration by parts formula related to historical processes as far as Proposition 4 in §8 is valid. Hence, combining the above results, we obtain

$$\begin{aligned} \mathbf{P} \left\{ F(K) \int \int_{C(t)} \Psi(s, y) dM \right\} &= \lim_{l \rightarrow \infty} \mathbf{P} \left\{ (F \circ W[l])(K) \int \int_{C(t)} \Psi(s, y) dM \right\} \\ &= \lim_{l \rightarrow \infty} \mathbf{P} \int \int_{C(t)} Pr[F \circ W[l]] \gamma(s, y) \Psi(s, y) K_s(dy) ds \\ &= \mathbf{P} \int \int_{C(t)} Pr[F](s, y) \gamma(s, y) \Psi(s, y) K_s(dy) ds, \end{aligned}$$

which concludes Theorem 3.

## §9.2 Stochastic Integration by Parts: Proof of Theorem 2

Since the complete proof is longsome and tiresome, computation in details will be sacrificed for the sake of simplicity and clearness. The basic idea is due to §7 in [Dk99a].

Thanks to (A.1), it suffices to verify the integral formula for a special  $\{\gamma_N, a_N, b_n, 0\}$ -historical process  $K_{\cdot \wedge T_N}$  under  $\mathbf{Q}_N$  instead of the generalized  $K$  (GHP) with  $\mathbf{P}$ . Indeed, since  $d\mathbf{P} = \mathcal{E}(t \wedge T_N)^{-1} d\mathbf{Q}_N$ , what we have to show is as follows:

(The Modified Stochastic Integration By Parts Formula)

$$\begin{aligned} & \mathbf{Q}_N \left\{ \mathcal{E}(t \wedge T_N)^{-1} \cdot \Phi(K_{\cdot \wedge T_N}) \int \int_{C(t)} \Psi(s, y) dM(s, y) \right\} \\ &= \mathbf{Q}_N \left\{ \mathcal{E}(t \wedge T_N)^{-1} \int \int_{C(t)} Pr[\Phi](s, y) \gamma(s, y) \Psi(s, y) K_{s \wedge T_N}(dy) ds \right\}. \end{aligned}$$

Note that both sides above are well-defined by virtue of (A.4). Notice that Eq.(12)-(14) remains valid even for  $\varphi = \Psi \cdot \mathcal{E}^{-1}$ . Hence, by the arguments on exponential martingale formalism for the historical process,  $\Lambda[\Psi \cdot \mathcal{E}^{-1}](t)$  is a  $\mathcal{H}_t$ -martingale and the measure  $\mathbf{Q}_N[\Psi \cdot \mathcal{E}^{-1}]$  is given by  $\mathbf{Q}_N[\{\cdot\} \Lambda[\Psi \cdot \mathcal{E}^{-1}]]$ . Then it follows from Dawson's Girsanov theorem (Proposition 3 in §8) that, for any positive  $\varepsilon$ ,

$$\mathbf{Q}_N\{\Phi(K_{\cdot \wedge T_N})\} = \mathbf{Q}_N[\varepsilon \Psi \mathcal{E}^{-1}]\{\Phi(K_{\cdot \wedge T_N}[\varepsilon \Psi \mathcal{E}^{-1}])\}.$$

Immediately,

$$\begin{aligned} & \mathbf{Q}_N \left\{ \Phi(K_{\cdot \wedge T_N}) \cdot (\Lambda[\varepsilon \Psi \mathcal{E}^{-1}](t) - 1) \right\} \\ &+ \mathbf{Q}_N \left\{ \left( \Phi(K_{\cdot \wedge T_N}[\varepsilon \Psi \mathcal{E}^{-1}]) - \Phi(K_{\cdot \wedge T_N}) \right) \cdot (\Lambda[\varepsilon \Psi \mathcal{E}^{-1}](t) - 1) \right\} \\ &= \mathbf{Q}_N \left\{ \Phi(K_{\cdot \wedge T_N}) - \Phi(K_{\cdot \wedge T_N}[\varepsilon \Psi \mathcal{E}^{-1}]) \right\}. \end{aligned}$$

For simplicity we denote by  $I_1$  (resp.  $I_2$ ) the first (resp. second) term at the left hand side of the above equality, and put

$I_3 =$  the right hand side with the minus sign.

Then we find that the convergence

$$\varepsilon^{-1} \cdot (\Lambda[\varepsilon \Psi \mathcal{E}^{-1}](t) - 1) \rightarrow \int \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y), \quad \mathbf{Q}_N - a.s. \quad (\varepsilon \rightarrow 0)$$

is true (cf. Lemma 8, [Dk99a]). Hence we readily obtain

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} I_1 = \mathbf{Q}_N \left\{ \Phi(K_{\cdot \wedge T_N}) \cdot \int \int_{C(t)} \Psi(s, y) \mathcal{E}(t \wedge T_N)^{-1} dM(s, y) \right\}.$$

Paying attention to the fact that

$$\lim_{\varepsilon \downarrow 0} K^*[\varepsilon \Psi \mathcal{E}^{-1}; C](t) = 0, \quad \mathbf{Q}_N - a.s.,$$

we can show that  $\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} I_2 = 0$ , as well.

It remains to treat the third term  $I_3$ . In order to discuss the convergence of  $I_3$  divided by  $\varepsilon$ , we need the following:

**Key Lemma** (cf. Lemma 12, [Dk99a])

$$\begin{aligned} \mathbf{Q}_N \int \int \left\{ \Phi(\mathcal{M}[s, y] K_{\cdot \wedge T_N}) - \Phi(K_{\cdot \wedge T_N}) \right\} \Lambda_{\Psi, \mathcal{E}^{-1}}(ds \otimes dy) \\ = - \mathbf{Q}_N \int \int Pr[\Phi] \gamma(s, y) \Psi(s, y) \mathcal{E}^{-1}(t \wedge T_N) dK_{s \wedge T_N}(y) ds. \end{aligned}$$

On the other hand, for  $\varepsilon > 0$  we have

$$\begin{aligned} \mathbf{Q}_N [\Phi(K[\varepsilon \varphi]) - \Phi(K)] / \mathcal{F} \\ = \varepsilon \cdot e^{-\varepsilon \Lambda_\varphi((\tau, \infty) \times C)} \int \int_{C(\infty)} \{ \Phi(\mathcal{M}[s, y] K) - \Phi(K) \} \Lambda_\varphi(ds \otimes dy) + R(\varepsilon, \Phi, \varphi) \end{aligned} \quad (18)$$

where the residue function  $R$  satisfies  $|R(\varepsilon, \Phi, \varphi)| \leq o(\varepsilon)$ . From (18) we get the convergence

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} I_3 = -\mathbf{Q}_N \int \int_{C(t)} Pr[\Phi] \gamma(s, y) \cdot \Psi \mathcal{E}^{-1} dK_{s \wedge T_N} ds. \quad (19)$$

In fact, a simple application of the above-mentioned Key Lemma yields the required result. To complete the proof, we have only to combine the above results.

### §9.3 Cluster Representation Argument: Proof of Key Lemma

For the proof of Key Lemma, although it is very technical, we are based on the cluster representation argument [D93] (see also [DP91]). For the details, we refer to the arguments stated in §8 in [Dk99a]. The following lemmas are merely essential parts of the discussion.

For any  $y \in C^s$ ,  $R(s, t, y)$  denotes the canonical measure (cf §5) in the theory of cluster random measures (e.g. [D93], [DP91]). Actually,  $R$  is a  $\sigma$ -finite measure such that

$$R(s, t, y; M_F(C)) = r_{s,t}.$$

Here the crucial point is that the total mass  $r_{s,t}$  does not depend on  $y$ . So  $r_{s,t}^{-1} dR(s, t, y)$  becomes a probability measure. It is interesting to note that  $K_t$  is a sum of independent nonzero clusters with laws  $r_{s,t}^{-1} R(s, t, y; dh)$ , conditional on  $L[s, t]$  (see §6). Furthermore, conditional on  $\mathcal{F}_s$ ,  $L[s, t]$  can be regarded as a Poisson point process with intensity  $r_{s,t} \gamma(s) K_s$ . This is one of the most important points for the computation in terms of clusters growing from the points of  $L[s, t_{l+1}]$  in what follows. We define a measure  $S$  by the following equation: for  $\forall g \in b\mathcal{B}([M_F(C)]^{k-l} \rightarrow \mathbf{R})$ ,

$$\begin{aligned} \int g(\eta_{l+1}, \dots, \eta_k) S_{s,y}(d\eta_{l+1} \otimes \dots \otimes d\eta_k) \\ = \int g(h(t_{l+1}), \dots, h(t_k)) \cdot \mathbf{I}\{h(t_{l+1}) \neq 0\} Q(s, y; dh) \end{aligned}$$



where  $Q(s, y; dh)$  is a  $\sigma$ -finite measure on  $C(M_F(C))$  (cf. Eq.(7) in §5).  $S_{s,y}^*$  is the normalization of  $S_{s,y}$ , given by  $dS_{s,y}^* := r_{s,t_{l+1}}^{-1} dS_{s,y}$ . Moreover, we define

$$\begin{aligned} \Xi(s; E) &:= \int \int \cdots (k-l) \cdots \int \varphi(K(t_1), \dots, K(t_l), \sum_{i=1}^m \eta_{l+1}^i, \dots, \sum_{i=1}^m \eta_k^i) \\ &\quad \times \bigotimes_{i=1}^m S_{s,y}^*(d\eta_{l+1}^i \otimes \cdots \otimes d\eta_k^i), \end{aligned}$$

where  $E = \{y_1, \dots, y_m\} (\neq \emptyset)$ .

Take the mass  $\varphi$  as  $(\Psi\mathcal{E}^{-1})(s, y)$  at each point  $y$  (cf. §6). For simplicity we set

$$\Delta[\Phi](\mathcal{M}; s, y, K) := \Phi(\mathcal{M}[s, y]K_{\cdot \wedge T_N}) - \Phi(K_{\cdot \wedge T_N}).$$

Recall the assumption (A.3). Immediately we can get

$$\begin{aligned} \mathbf{Q}_N &\int \int_{C(\infty)} \Delta[\Phi](\mathcal{M}; s, y, K) \Lambda_{\Psi\mathcal{E}^{-1}}(ds \otimes dy) \\ &= \mathbf{Q}_N \int_{a+}^b \int_C \Delta[\Phi](\mathcal{M}; s, y, K) \lambda_s[\Psi\mathcal{E}^{-1}](dy) ds \\ &= \int_{a+}^b ds \mathbf{Q}_N \left\{ \sum_{y \in L[s, u]} \Delta[\Phi](\mathcal{M}; s, y, K) \cdot (\Psi\mathcal{E}^{-1})(s, y) \right\}. \end{aligned}$$

In the following calculation, we may take much advantage of those concepts such as i) the Markov property of  $K_t$ ; ii) the infinite divisibility of the law of historical process; iii) the Poisson nature of the location  $L[s, t_{l+1}]$ . Hence we can proceed with the computation. In fact,

$$\begin{aligned} &\mathbf{Q}_N \left\{ \sum_{y \in L[s, u]} \Delta[\Phi](\mathcal{M}; s, y, K) \cdot (\Psi\mathcal{E}^{-1})(s, y) \right\} \\ &= \mathbf{Q}_N \left\{ \mathbf{P} \left[ \sum_{y \in L[s, u]} \mathbf{P}\{\Delta[\Phi] \cdot \Psi\mathcal{E}^{-1} | \mathcal{F}_s \vee \sigma(L[s, u])\} \middle| \mathcal{F}_s \right] \right\} \\ &= \mathbf{Q}_N \left\{ \mathbf{P} \left[ \sum_{y \in L[s, u]} \{\Xi(s; L[s, u] \setminus \{y\}) - \Xi(s; L[s, u])\} \cdot \Psi\mathcal{E}^{-1} \middle| \mathcal{F}_s \right] \right\} \quad (20) \end{aligned}$$

It is easy to see the following lemma.

**Lemma 5** *The last expression of (20) is equivalent to*

$$\begin{aligned} &\mathbf{Q}_N \int_C (\Psi\mathcal{E}^{-1})(s, y) \cdot r_{s,t_{l+1}} \gamma(s, y) K_{s \wedge T_N}(dy) \left[ \exp(-r_{s,t_{l+1}} K_s(C)) \cdot \right. \\ &\quad \times \sum_{m=0}^{\infty} \frac{1}{m!} \int \int \cdots (m) \cdots \int_{[C]^m} \{\Xi(s; \{y_1, \dots, y_m\}) - \Xi(s; \{y_1, \dots, y_m, y\})\} \cdot \\ &\quad \times (r_{s,t_{l+1}})^m K_s^{\otimes m}(dy_1, \dots, dy_m) \left. \right]. \end{aligned}$$

A simple computation implies that the integral expression in Lemma 5 is also equal to

$$\begin{aligned} \mathbf{Q}_N & \int_C (\Psi \mathcal{E}^{-1})(s, y) \gamma(s, y) K_{s \wedge T_N}(dy) \cdot \left[ \int \int \cdots (k-l) \cdots \int_{[M_F(C)]^{k-l}} \right. \\ & \times \mathbf{P}\{\varphi(K(t_1), \dots, K(t_k)) - \varphi(K(t_1), K(t_l), K(t_{l+1}) + \eta_{l+1}, \dots, K(t_k) + \eta_k) | \mathcal{F}_s\} \\ & \times r_{s, t_{l+1}} \cdot S_{s, y^{s-}}^* (d\eta_{l+1} \otimes \cdots \otimes d\eta_k) \Big]. \end{aligned} \quad (21)$$

While, taking (7), (8) in §5, the Campbell measure theory, and predictable section argument into consideration, we readily obtain

**Lemma 6** *The following equality holds for all  $s, y$ :*

$$\begin{aligned} Pr [\Phi](s, y) &= \int \int \cdots (k-l) \cdots \int r_{s, t_{l+1}} \cdot S_{s, y^{s-}}^* (d\eta_{l+1} \otimes \cdots \otimes d\eta_k) \cdot \\ & \times P\{\varphi(K(t_1), \dots, K(t_l), K(t_{l+1}) + \eta_{l+1}, \dots, K(t_k) + \eta_k) - \varphi(K(t_1), \dots, K(t_k)) | \mathcal{F}_s\}. \end{aligned}$$

Therefore, an application of the above assertion with Lemma 5 implies

$$\begin{aligned} & - \mathbf{Q}_N \int \int_{C(t)} Pr[\Phi](\gamma \cdot \Psi \mathcal{E}^{-1})(s, y) dK_{s \wedge T_N} ds \\ &= \int_{\tau+}^t ds \left\{ \mathbf{Q}_N \int_C (-Pr[\Phi]) \gamma \cdot \Psi \mathcal{E}^{-1} dK_{s \wedge T_N} ds \right\} = \int_{\tau+}^t Eq.(21) ds = \int_{\tau+}^t Eq.(20) ds \\ &= \mathbf{Q}_N \int \int_{C(t)} \Delta[\Phi](\mathcal{M}; s, y, K) \Lambda_{\Psi \mathcal{E}^{-1}}(ds \otimes dy), \end{aligned}$$

which completes the proof.

## 10 Evans-Perkins Type Formula: Proof of Theorem 4

Since  $\mathbf{P}[K_t(C)^2]$  is uniformly bounded on compact intervals, our major premise guarantees the finiteness of the quantity  $\mathbf{P}[F(K)^2]$ . Therefore we can apply Theorem 1 (§3) for  $F(K)$  to obtain that

$$F(K) = \mathbf{P}[F(K)] + \int_{\tau}^{\infty} \int_C f(s, y) dM^K(s, y), \mathbf{P} - a.s. \quad (22)$$

holds for some  $f$  in  $L_{\infty}^2(K, \mathbf{P})$ . While, it follows from the covariance formula in the theory of stochastic integration that

$$\begin{aligned} \mathbf{P} & \left[ \left( \int \int_{C(\infty)} f(s, y) dM^K(s, y) \right) \left( \int \int_{C(t)} \Psi(s, y) dM^K(s, y) \right) \right] \\ &= \mathbf{P} \left[ \int_{\tau}^t \int_C f(s, y) \Psi(s, y) \gamma(s, y) K_s(dy) ds \right] \end{aligned} \quad (23)$$

for all  $t > \tau$  and  $\Psi$  in  $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$ . Rewriting the left hand side of Eq.(23) we get

$$\mathbf{P} \left[ F(K) \int_{\tau}^t \int_C \Psi(s, y) dM^K(s, y) \right] \quad (24)$$

by employing the predictable representation property (22). Hence we may apply Theorem 3 (§7) to rewrite (24), because the stochastic integration by parts formula is valid for any bounded  $(\mathcal{C}_t \times \mathcal{F}_t)$ -predictable functions. So that, from (23)

$$\mathbf{P} \int \int_{C(t)} f(s, y) \Psi(s, y) \gamma(s, y) dK_s ds = \mathbf{P} \int \int_{C(t)} Pr[F](s, y) \Psi(s, y) \gamma(s, y) dK_s ds.$$

On this account, the general theory of Hilbert spaces shows that

$$\mathbf{P} \int_{\tau}^t \int_C \{f(s, y) - Pr[F](s, y)\}^2 \gamma(s, y) K_s(dy) ds = 0.$$

Therefore the uniqueness argument allows us to conclude that  $\int \int_{C(t)} f dM$  is equivalent to  $\int \int_{C(t)} Pr[F] dM$ ,  $\mathbf{P}$ -a.s. Note that  $Pr[F](s, y)$  become null for  $K_s$ -a.s.  $y$ , for any  $s > t$ , by its construction, as long as we choose  $t$  largely enough for the support of  $m$  to be contained in  $[\tau, t]$ . Consequently, the above integral  $\int \int Pr[F] dM$  can be replaced by  $\int \int_{C(\infty)} Pr[F] dM$ , which completes the proof. This goes quite similarly as in the proof of Theorem 2.5 in [EP95].

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